Compressive Spectral Video by Optimal 4D-Sphere Packing


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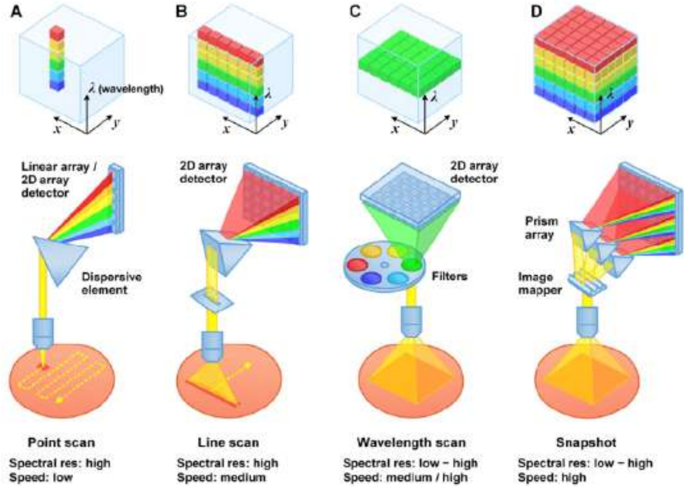
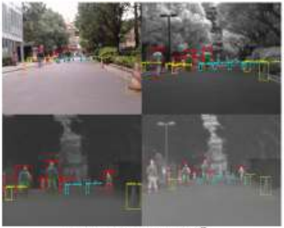
Traditional Approaches to Capture Spectral Images



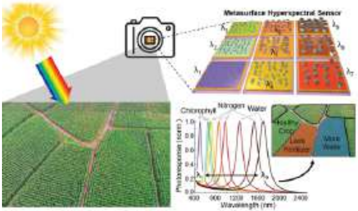
[1]

[1] Wang, Y. W., Reder, N. P., Kang, S., Glaser, A. K., and Liu, J. T., "Multiplexed Optical Imaging of Tumor-Directed Nanoparticles: a Review of Imaging Systems and Approaches. Nanotheranostics", Ivyspring International Publisher, 1(4), 2017.

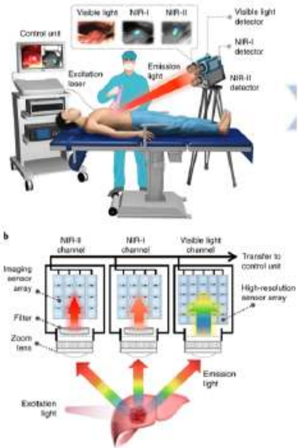
Spectral Video Applications



Autonomous cars[2]



Smart farming[3]



Guided surgery[4]

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[2] K. Takumi, K. Watanabe, Q. Ha, A. Tejero-De-Pablos, Y. Ushiku, and Tatsuya Harada., "Multispectral Object Detection for Autonomous Vehicles," in In Proceedings of the on Thematic Workshops of ACM Multimedia, 2017.

[3] Hu, Z., Fang, C., Li, B. et al., "First-in-Human Liver-Tumour Surgery Guided by Multispectral Fluorescence Imaging in the Visible and Near-infrared-I/II Windows," in Nat Biomed Eng, vol. 4, pp. 259–271, 2020.

[4] Jon W. Stewart, Jarrett H. Vella, Wei Li, Shanhui Fan, and Maiken H. Mikkelsen, "Ultrafast Pyroelectric Photodetection with On-Chip Spectral Filters," in Nature Materials, 2019, DOI: 10.1038/s41563-019-0538-6.

What is Sphere Packing?

The sphere packing problem asks for the densest packing of $\mathbb{R}^n$ with congruent balls. Equivalent to answer the question:

What is the largest fraction of $\mathbb{R}^n$ that can be covered by congruent balls with disjoint interiors?

Sphere Packing Density

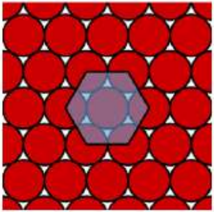
One-dimensional sphere packing is boring:

1D

density=1

Two-dimensional sphere packing is more interesting and attractive

2D



density=$\frac{\pi}{\sqrt{12}} \approx 0.9068$

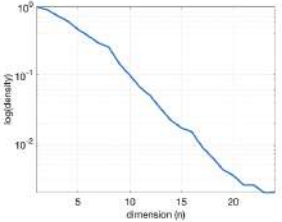
Three dimensions strains human ability to prove

3D

density=$\frac{\pi}{\sqrt{18}} \approx 0.7404$

What happens in four dimensions?

Sphere Packing Density in $\mathbb{R}^n$ and Applications



n	density	n	density	n	density	n	density
1	1.00000	7	0.29529	13	0.03201	19	0.00412
2	0.90689	8	0.25366	14	0.02162	20	0.00339
3	0.74048	9	0.14577	15	0.01685	21	0.00246
4	0.61685	10	0.09961	16	0.01470	22	0.00245
5	0.46525	11	0.06623	17	0.00881	23	0.00190
6	0.37294	12	0.04945	18	0.00616	24	0.00192

Sphere packing density in $\mathbb{R}^n$ [5]. Optimal density in blue color.

Applications in Computational Imaging

* Compressive video[6], $f(x,y,t) \in \mathbb{R}^3$ and compressive spectral imaging[7, 8] $f(x,y,\lambda) \in \mathbb{R}^3$ are sampling problems in 3D.
* Compressive spectral video $f(x,y,\lambda,t) \in \mathbb{R}^4$ are sampling problems in 4D.

[5] H. Cohn, "A Conceptual Breakthrough in Sphere Packing", Notices of the American Mathematical Society, Vol. 64, pp.102-115, 2017.

[6] E. Vera; F. Guzman; N. Diaz, "Shuffled Rolling Shutter for Snapshot Temporal Imaging", Opt. Express, Vol. 30, pp.887-901,2022.

[7] N. Diaz, A. Alvarado, P. Meza, F. Guzmán and E. Vera, "Multispectral Filter Array Design by Optimal Sphere Packing," in IEEE Transactions on Image Processing, vol. 32, pp. 3634-3649, 2023, doi: 10.1109/TIP.2023.3288414.

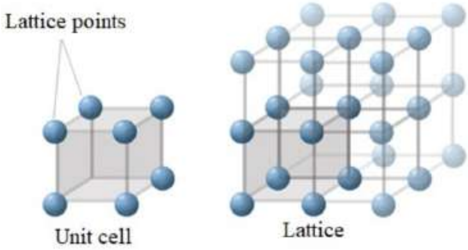
[8] A. Alvarado; N. Díaz; P. Meza; F. Guzman, E. Vera; , "Multispectral Mosaic Design using a Sphere Packing Filter Array", in imaging and Applied Optics Congress 2022.

What is a Lattice?

A typical lattice $\mathbf{\Lambda} \in \mathbb{R}^n$ thus has the form

$$\mathbf{\Lambda} = \sum_{i=1}^{n} a_i v_i | a_i \in \mathbb{Z} \tag{1}$$

* where $\mathbf{M} = [v_1, \ldots, v_n]$ is a unit cell or Generator Matrix basis in $\mathbb{R}^n$
* The Gram Matrix $\mathbf{A} = \mathbf{M}^T\mathbf{M}$. Its entries (i, j) are given by $\langle v_i, v_j \rangle$.



$\mathbf{M}$ is a unit cell and $\mathbf{\Lambda}$ is a lattice.

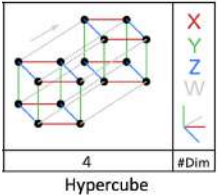
4D-Lattice: Generator Matrix and Gram Matrix

The unit cell in $\mathbb{R}^4$ is $\mathbf{M}_4$ lattice has generator matrix:

$$\mathbf{M}_4 = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The $\mathbf{M}_4$ lattice has Gram Matrix:

$$\mathbf{A}_4 = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$



Compute the Density of a Lattice

Density of a lattice in a unit cell:

$$\frac{\text{Vol}(B_r^n)}{\text{Vol}(\mathbb{R}^n/\Lambda)} = \frac{\frac{\pi^{n/2}}{(n/2)!} r^n}{\sqrt{\det(\mathbf{A})}} \tag{2}$$

* n-dimensional Sphere's volume:
  $\text{Vol}(B_r^n) = \frac{\pi^{n/2}}{(n/2)!} r^n$, where $(n/2)!$ means $\Gamma(n/2+1)$.
* n-dimensional Lattice volume:
  $\text{Vol}(\mathbb{R}^n/\Lambda) = \sqrt{\det(\Lambda)} = \sqrt{\det(\mathbf{A})} = \det(\mathbf{M})$
* n-dimensional radius:
  Let $r = N(\Lambda)$ denote the minimal non-zero value of $\langle v, v \rangle$ among all $v \in \Lambda$.

n	1	2	3	4	5	6	7	8	24
Λ	A_1	A_2	A_3	D_4	D_5	E_6	E_7	E_8	Leech
due to		Lagrange	Gauss	Korkine-Zolotareff		Blichfeldt			Cohn-Kumar

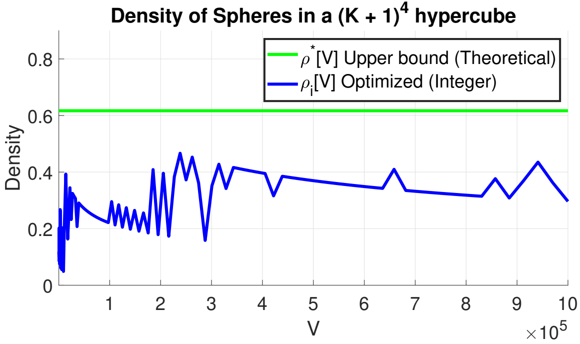
[5] Conway, J. and Sloane, N.J.A, "Sphere Packings, Lattices and Groups", Springer New York, 2013.

Best Packing Known in $\mathbb{R}^4$: M_4 Lattice and Upper Bound

In particular, for the $\mathbf{M}_4$ lattice the SP density corresponds to

$$\frac{\text{Vol}(B_r^4)}{\text{Vol}(\mathbb{R}^4/\mathbf{M}_4)} = \frac{\frac{\pi^2}{2}r^4}{\sqrt{\det(\mathbf{A}_4)}} = 0.61685\ldots, \tag{3}$$

where $r_{\mathbf{M}_4} = \Phi(\mathbf{M}_4) = \frac{1}{\sqrt{2}}$ is the radius of the best known $4D$-SP. The following section shows how to use 4D-SP for sampling spectral-video.

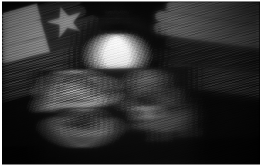


Discrete Model

The corresponding discrete model is as follows:

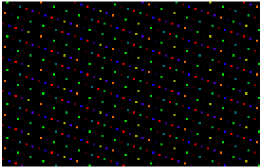
$$\mathbf{Y} = \sum_{t=0}^{T-1} \sum_{k=0}^{K-1} \mathcal{X}_{(:,:,k,t)} \odot \mathcal{C}_{(:,:,k,t)} + \mathbf{\Omega}, \tag{4}$$

where $\mathcal{X} \in \mathbb{R}^{M \times N \times K \times T}$ is the tensor that represents the 4-dimensional spectral-video datacube, and $\mathcal{C} \in \mathbb{R}^{M \times N \times K \times T}$ denotes the tensor of the $4D$-CA.



measurement $\mathbf{Y}_{(:,:)}$

spectral-video $\mathcal{X}_{(:,:,:,t)}$

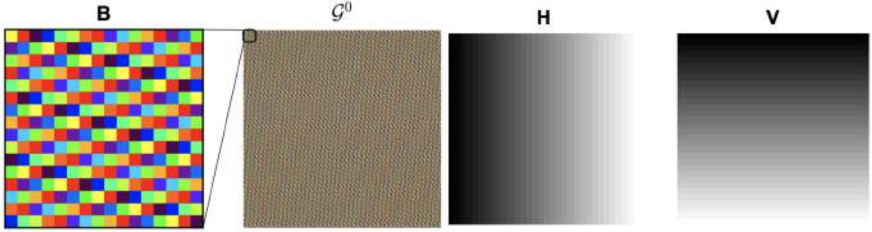


coded aperture $\mathcal{C}_{(:,:,:,t)}$

Multispectral Filter Array by Optimal Sphere Packing

The sampling of spectral-video can leverage from the following solution to 3DN^2QP[6] to place the spheres within a 3D-container, $\mathbf{B}$ as follows:

$$\mathbf{B} = ((a \odot \mathbf{V} + b \odot \mathbf{H}) \mod K + 1), \tag{5}$$



[6] Allison, Lloyd and Yee, CN and McGaughey, M, "Three-Dimensional Queens Problems", Monash University, Department of Computer Science, 1989.

4D-Coded Aperture (CA) Sphere Packing Design

The positions of MSFA-OSP at the t^{th} frame are given by

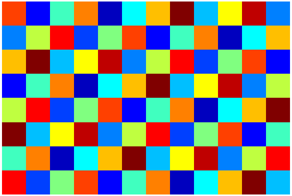
$$\mathcal{G}_{(:,:,0)} = \mathbf{A} \otimes \mathbf{B}, \tag{6}$$

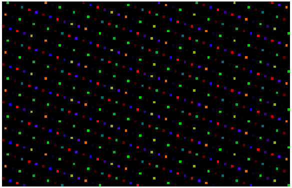
where $\mathbf{A}$ is a matrix of all ones such that $\mathbf{A} \in \{1\}^{\alpha \times \beta}$, where $\alpha = \lfloor \frac{M}{K} \rfloor$, and $\beta = \lfloor \frac{N}{K} \rfloor$. The successive t^{th} frame is computed by permuting the tensor $\mathcal{G}_{(:,:,t-1)}$

$$\mathcal{G}_{(:,:,t)} = ((\mathcal{G}_{(:,:,t-1)} + c) \mod K + 1), \tag{7}$$

where c is an integer constant that permutes $\mathcal{G}_{(:,:,0)}$ along time dimension. The multispectral pattern $\mathcal{G}$ can be reorganized as CA

$$\mathcal{C}_{(i,j,k,t)} = \begin{cases} 1 & \text{if } k = \mathcal{G}_{(i,j,t)} \\ 0 & \text{if } k \neq \mathcal{G}_{(i,j,t)}, \end{cases} \tag{8}$$

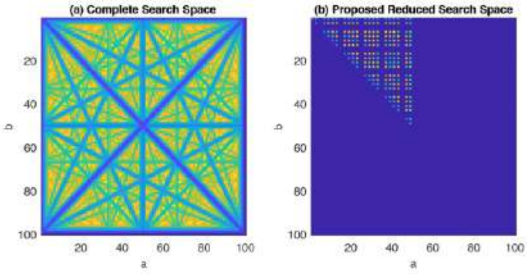

MSFA $\mathcal{G}_{(:,:,t)}$.


coded aperture $\mathcal{C}_{(:,:,:,t)}$.

Compute Spheres Distance

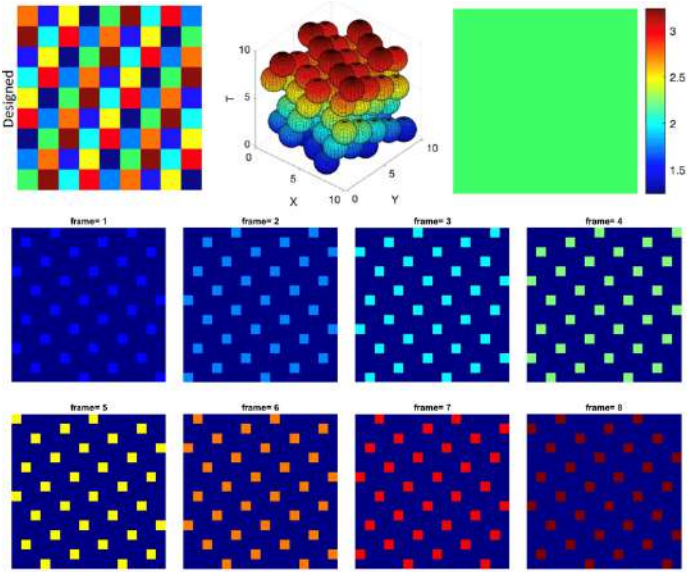
The resulting tensor $\mathcal{G} \in \mathbb{R}^{M\times N\times T}$ can be reorganized as $\mathbf{p}_l = [i, j, \mathcal{G}_{(i,j,t)}, t]$, where $\mathbf{P} = [\mathbf{p}_1, \dots \mathbf{p}_l \dots \mathbf{p}_V] \in \mathbf{R}^{4\times V}$, with indexes $i, j \in \{1, \dots, K\}$ and $k \in \{1, \dots, K\}$, where $V = K^3$ is the number of spheres. Thus, the distance function of V spheres is

$$d^*(V) = \max(\min_{1\leq l_1 < l_2 \leq V}, D_{l_1, l_2}), \tag{9}$$



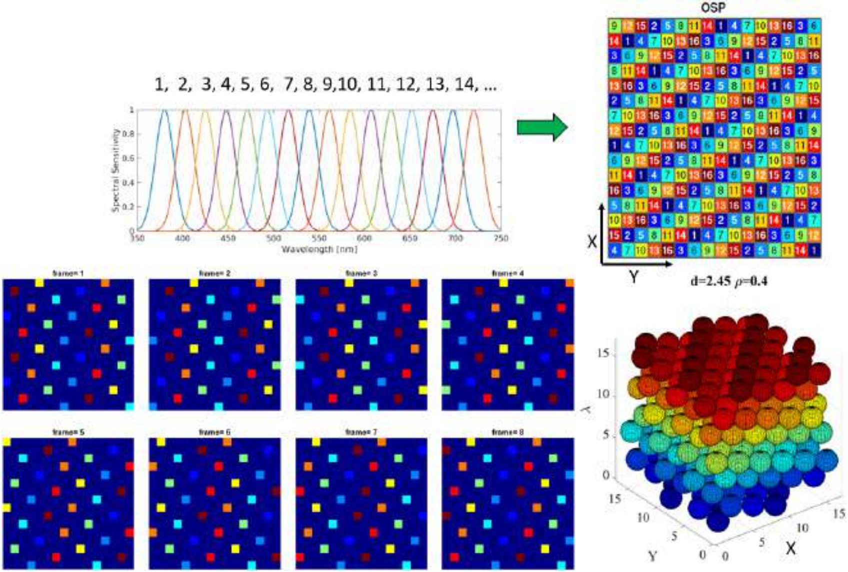
Coded Aperture Design (Step 1)

First, we solve the temporal dimension



Coded Aperture Design (Step 2)

Then, we assign filters to each frame

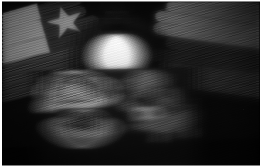


Reconstruction Algorithm

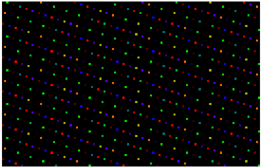
We start by expanding the measurement $\mathbf{Y}$ into the datacube $\bar{\mathcal{X}}_{(:,:,k,t)}$ by using the CA $\mathcal{C}_{(:,:,k,t)}$ such that

$$\bar{\mathcal{X}}_{(:,:,k,t)} = \mathcal{C}_{(:,:,k,t)} \odot \mathbf{Y}. \tag{10}$$

spectral-video $\hat{\mathcal{X}}_{(:,:,:,t)}$.



measurement $\mathbf{Y}_{(:,:)}$.

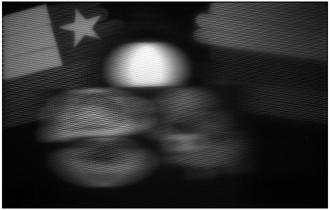


coded aperture $\mathcal{C}_{(:,:,:,t)}$.

The algorithm to recover the underlying datacube is Nearest Neighbor Interpolation (NNI)[7], whose input is $\bar{\mathcal{X}}_{(:,:,:,t)}$ and its output is $\hat{\mathcal{X}}_{(:,:,:,t)}$.

[7] Amidror, Isaac, "Scattered Data Interpolation Methods for Electronic Imaging Systems: a Survey.", J. Electronic Imaging, Vol. 11, pp.157-176, 2002.

Comparison of Image Quality Reconstruction



From a single snapshot $\mathbf{Y}$, we are able to recover a spectral-video with 16 frames and 16 bands.

Spectral-video Groundtruth

$\mathcal{X} \in \mathbb{R}^{1220\times775\times16\times16}$.

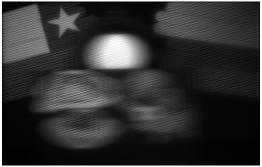
Spectral-video reconstruction $\hat{\mathcal{X}}$.

PSNR 31.42 dB, SSIM 0.94, SAM 0.07

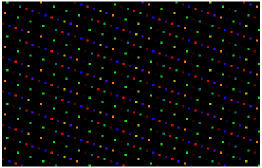
Conclusions

* We introduced a novel compressive spectral-video sensing approach that exploits optimal sphere packing.
* Our approach is able to accurately recover a spectral video from a single snapshot.
* The proposed approach obtains image reconstruction quality up to 31.42 [dB] of PSNR and 0.07 of SAM.

Spectral-video reconstruction $\hat{\mathcal{X}}_{(:,:,:,t)}$.



measurement $\mathbf{Y}_{(:,:)}$.



coded aperture $\mathcal{C}_{(:,:,:,t)}$.

Future Work

* Compressive spectral depth $f(x, y, z, \lambda) \in \mathbb{R}^4$.
* Compressive spectral light field samples a function $f(x, y, z, \theta, \phi) \in \mathbb{R}^5$, is a problem in 5D.
* Sampling the plenoctic function involves sensing in 7D $f(x, y, z, \theta, \phi, \lambda, t) \in \mathbb{R}^7$ being (x, y, z) 3D-space dimensions, (λ) spectral dimension, (θ, ϕ) two angular dimensions, and (t) time.

n	density	n	density	n	density	n	density
1	1.00000	7	0.29529	13	0.03201	19	0.00412
2	0.90689	8	0.25366	14	0.02162	20	0.00339
3	0.74048	9	0.14577	15	0.01685	21	0.00246
4	0.61685	10	0.09961	16	0.01470	22	0.00245
5	0.46525	11	0.06623	17	0.00881	23	0.00190
6	0.37294	12	0.04945	18	0.00616	24	0.00192

Table 1: SP densities in $\mathbb{R}^n$ with $1 \leq n \leq 24$[8].

[8] H. Cohn, "A Conceptual Breakthrough in Sphere Packing", Notices of the American Mathematical Society, Vol. 64, pp.102-115, 2017.

Funding

ANID ANILLOS ATE220022

ANID FONDECYT 1201081

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History of the Sphere Packing Problem

- 1611
- Johannes **Kepler conjectured** about the closest packing of equal spheres.
- He did not have a prove to the conjecture.

- 1998
- Thomas Hales provides the formal proof of Kepler's conjecture.
- But **eliminating all posible irregular arregements** is very difficult, and this is what made the Kepler conjecture so hard to prove.

- 1831
- Carl Friedrich Gauss proved that the highest packing fraction that can be achieved by any packing of equal sphere.
- He proved that the Kepler conjecture is true if the spheres have to be arranged in a **regular lattice.**

- 2017
- Maryna Viazovska solved sphere packing problem in **8-dimensions** [1] (E_8 lattice). And in collaboration with others **24-dimensions** [2] (Leech lattice).
- **Winner of the Fields Medal 2022.**

Face-centered cubic structure

Theorem 1: No packing of congruent balls in Euclidean three space has density greater than that of the **face-centered cubic packing**, which corresponds to:

$$\rho = \frac{\pi}{3\sqrt{2}} \approx 0.7405$$

[1] M. S. Viazovska, "The sphere packing problem in dimension 8,"Annals of Mathematics, vol. 185, no. 3, pp. 991-1015, 2017.
[2] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska, "The sphere packing problem in dimension 24," Annals of Mathematics, vol. 185, no. 3, pp. 1017-1033, 2017.

Face-centered Cubic Lattice

Definitions:

The generator matrix $\mathbf{M}$ has $v_1, \ldots, v_n$.

The Gram matrix $\mathbf{A} = \mathbf{M}^T\mathbf{M}$. Its entries (i, j) are given by $\langle v_i, v_j \rangle$.

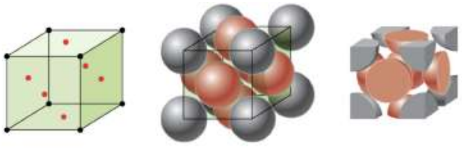
The face-centered cubic (FCC) lattice has generator matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Example: The FCC has Gram matrix:

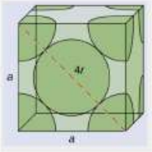
$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Face-Centered Cubic (FCC) Density



Face-centered cubic structure

Figure 3. A *face-centered cubic* solid has atoms at the corners and, as the name implies, at the centers of the faces of its unit cells.

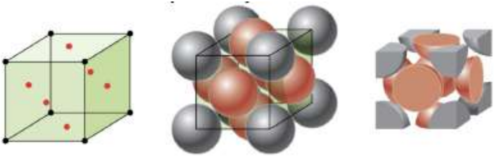


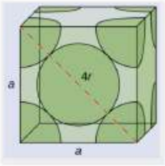
Let $r = N(A_3) = \frac{\sqrt{2}}{2}$ and $n = 3$

$$\frac{\mathsf{Vol}(B_r^n)}{\mathsf{Vol}(\mathbb{R}^n/A_3)} = \frac{\frac{\pi^{n/2}}{(n/2)!}r^n}{\sqrt{\det(\mathbf{A})}} = \frac{\frac{4\pi r^3}{3}}{2} = 0.74 \tag{11}$$

FCC include aluminium, copper, gold and silver.

Face-Centered Cubic: Geometrical Calculations





Face-centered cubic structure

Figure 3. *A face-centered cubic solid has atoms at the corners and, as the name implies, at the centers of the faces of its unit cells.*

$$V_{\text{FCC spheres}} = \left(8 \cdot \tfrac{1}{8} + 6 \cdot \tfrac{1}{2}\right) \frac{4\pi}{3} r^3 \quad (1)$$

$$= \tfrac{16}{3} \pi r^3. \quad (2)$$

The diagonal of a face of the unit cell is $4\,r$, so each side is of length $2\sqrt{2}\,r$. The volume of the unit cell is therefore

$$V_{\text{FCC unit cell}} = \left(2\sqrt{2}\,r\right)^3 = 16\sqrt{2}\,r^3, \quad (3)$$

giving a packing density $\eta = V_{\text{spheres}} / V_{\text{cell}}$ of

$$\eta_{\text{FCC}} = \frac{\pi}{3\sqrt{2}} \quad (4)$$

$$= 0.74048 \ldots \quad (5)$$